UNDULAR FLOW OF AN EVAPORATING VISCOUS LIQUID FILM UNDER UNIFORM INJECTION

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The wave profile on the free surface of a thin viscous liquid film flowing along a porous surface under uniform injection is established here by the asymptotic method which has been developed in [1, 2].

In view of the growing number of technical applications, much attention is paid since recently to the study of flowing thin layers (films) of viscous incompressible fluids [3-9]. Under actual conditions at rather low flow rates already, one notes undular flow modes in the film as its free surface acquires the shape of a periodic wave with a small amplitude [10, 11]. Injection may change the shape of the free film surface, which in turn affects the processes of heat and mass transfer. It has been shown in [3], for instance, that during laminar film condensation the heat transfer rate and the condensation rate both increase during suction.

We consider a viscous liquid flowing down along a porous wall ($y = 0$) inclined at an angle α to the horizontal, through which quantities of the same liquid are injected uniformly at a velocity W (Fig. 1). The flow, which occurs due to gravity, is assumed steady along the x-axis. In a certain system of reference coordinates moving at a velocity U, then, the shape of the free surface $y = f(x)$ will remain invariable. The flow rate of the liquid through any section of the film is also assumed constant.

With the flow rate Q and the mean depth of liquid H as the characteristic quantities, the flow of liquid can be described by the following dimensionless parameters: the Reynolds number $Re = Q/\nu$ (ν denoting the kinematic viscosity of the medium), the Froude number $Fr^{-2} = gh^3/Q^2$ (g denoting the acceleration of gravity), and the velocity $c = UH/Q$.

In the system of coordinates moving relative to the porous channel wall $y = 0$ at the dimensionless velocity c we write the system of dimensionless equations of motion with respect to the flow function ψ ($u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$ denoting the respective velocity components along y and x) as follows:

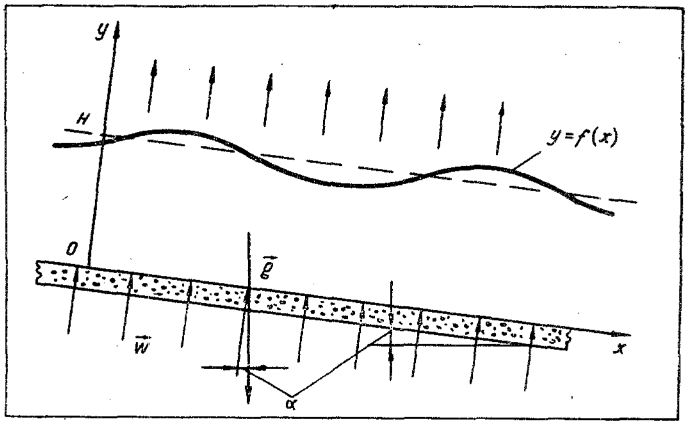

Fig. 1. Schematic diagram of a flowing thin viscous liquid film and transverse injection.

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$$\frac{D\left(\frac{\partial\psi}{\partial y},\psi\right)}{D(x,y)}=-\frac{\partial P}{\partial x}+\frac{1}{\mathrm{Fr}^2}\sin\alpha+\frac{1}{\mathrm{Re}}\cdot\frac{\partial}{\partial y}\Delta\psi, \tag{1}$$

$$\frac{D\left(\frac{\partial\psi}{\partial x},\psi\right)}{D(x,y)}=-\frac{\partial P}{\partial y}-\frac{1}{\mathrm{Fr}^2}\cos\alpha-\frac{1}{\mathrm{Re}}\cdot\frac{\partial}{\partial x}\Delta\psi. \tag{2}$$

Eliminating the pressure P from Eqs. (1) and (2) yields

$$\frac{D(\Delta\psi,\psi)}{D(x,y)}=\frac{1}{\mathrm{Re}}\Delta\Delta\psi. \tag{3}$$

On the free surface of the liquid $y = f(x)$ the tangential stresses must be zero

$$\frac{4f_x}{1+f_x^2}\cdot\frac{\partial^2\psi}{\partial x\partial y}+\left(\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}\right)\frac{1-f_x^2}{1+f_x^2}=0;\ f_x=\frac{df}{dx} \tag{4}$$

and the normal stresses must be zero

$$-P-\frac{2}{\mathrm{Re}}\cdot\frac{\partial^2\psi}{\partial x\partial y}\cdot\frac{1-f_x^2}{1+f_x^2}+\frac{2}{\mathrm{Re}}\left(\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}\right)\frac{f_x}{1+f_x^2}=0. \tag{5}$$

The boundary conditions at the porous wall are written as:

$$\left.\frac{\partial\psi}{\partial y}\right|_{y=0}=-c;\ -\left.\frac{\partial\psi}{\partial x}\right|_{y=0}=w. \tag{6}$$

The stipulation that the flow rate through any section of the film is constant can be expressed as

$$\int\limits_0^{f(x)}\frac{\partial\psi}{\partial y}dy=\psi[x,f(x)]-\psi[x,0]=q, \tag{7}$$

where $q = 1-c$ represents the dimensionless relative flow rate of liquid in the moving system of coordinates.

Proceeding now by the method of "narrow" bands [12, 13], we will consider the auxiliary problem of determining the flow function within the region occupied by a liquid at a fixed value of f(x). A change $x \to x/\varepsilon$, with the small parameter ε characterizing the "narrowness" of the film, will introduce this small parameter ε into all equations and boundary conditions here, making it feasible to seek the solution to Eq. (3) in the form of an asymptotic expansion with respect to this parameter:

$$\psi=\tilde{\psi}_0+\varepsilon\tilde{\psi}_1+\varepsilon^2\tilde{\psi}_2\equiv\psi_0+\psi_1+\psi_2. \tag{8}$$

Considering this expansion (8), we have for the zeroth approximation

$$\frac{\partial^4\psi_0}{\partial y^4}=0. \tag{9}$$

Conditions (4), (6), and (7) in the zeroth approximation will then be

$$\left.\frac{\partial^2\psi_0}{\partial y^2}\right|_{y=f(x)}=0;\ \left.\frac{\partial\psi_0}{\partial y}\right|_{y=0}=-c;\ -\left.\frac{\partial\psi_0}{\partial x}\right|_{y=0}=w;$$
$$\psi_0[x,f(x)]-\psi_0[x,0]=q. \tag{10}$$

Since we are interested here in waves with small amplitudes (i.e., $f(x) = 1 + \eta(x)$ with function $\eta(x)$ smaller than unity), hence the solution to Eq. (9) with boundary conditions (10) can be represented as

$$\psi_0=\frac{C_1}{6}y^3+\frac{C_2}{2}y^2-cy-wx+C_3, \tag{11}$$

where $C_1 = -3(c + q) + 3(2c + 3q)\eta - 9(c + 2q)\eta^2 + O(\eta^3)$; $C_2 = 3(c + q) - 3(c + 2q)\eta + 3(c + 3q)\eta^2 + O(\eta^3)$; and C_3 is an arbitrary constant. We note that the injection velocity w, as can be seen from solution (11) and boundary conditions (10), is a small quantity ($w = O(\varepsilon)$); in other words, we consider an undular flow mode in the case of thin films with a limited injection – not exceeding a specified rate.

For the first approximation we have

$$\frac{\partial^4\psi_1}{\partial y^4}=\mathrm{Re}\left[\frac{D\left(\frac{\partial^2\psi_0}{\partial y^2},\psi_0\right)}{D(x,y)}\right] \tag{12}$$

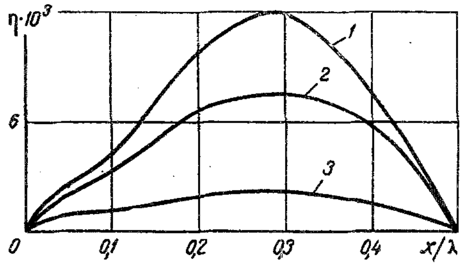

Fig. 2

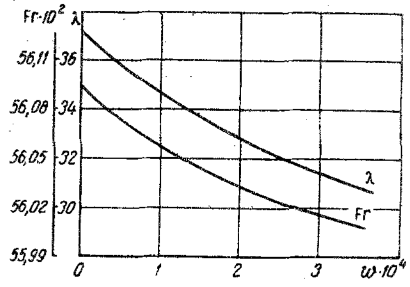

Fig. 3

Fig. 2. Shape of the wave profile as a function of the injection velocity w, for $c = 2.9$, $\mathrm{Re} = 23.6$, and $\alpha = 5°$; $w = 0$ (1), 10^{-3} (2), $2.5 \cdot 10^{-3}$ (3).

Fig. 3. Wavelength λ and the Froude number as functions of the injection velocity w, for $c = 2.9$, $\mathrm{Re} = 23.6$, and $\alpha = 5°$.

and the boundary conditions

$$\left.\frac{\partial^2\psi_1}{\partial y^2}\right|_{y=f(x)} = 0;\ \left.\frac{\partial\psi_1}{\partial y}\right|_{y=0} = 0;\ \left.\frac{\partial\psi_1}{\partial x}\right|_{y=0} = 0;\ \psi_1[x,\ f(x)] - \psi_1[x,\ 0] = 0, \tag{13}$$

wherefrom

$$\psi_1 = \frac{C_4}{6}\, y^3 + \frac{C_5}{2}\, y^2 + C_6 + \Pi_1(x,\ y), \tag{14}$$

with

$$C_4 = \frac{9}{8}\,\mathrm{Re}\, w\,[-c - q + (c + 2q)\,\eta - (c + 3q)\,\eta^2]$$

$$+ \frac{6}{35}\,\mathrm{Re}\,\eta_x[9q^2 + cq - c^2 + (c^2 - 2cq - 27q^2)\,\eta] + O(\eta^4);$$

$$C_5 = \frac{3}{8}\,\mathrm{Re}\, w\,(-c - q + q\eta - q\eta^2) - \frac{1}{35}\,\mathrm{Re}\,\eta_x[12q^2 + 6cq + c^2 - (24q^2 + 6cq)\,\eta] + O(\eta^4);$$

$\Pi_1(x,\ y)$ is the particular solution to the nonhomogeneous Eq. (12) (not shown here because of its unwieldiness), $\eta_x \equiv d\eta/dx$, and C_6 is an arbitrary constant.

For the second approximation we have

$$\frac{\partial^4\psi_2}{\partial y^4} = \mathrm{Re}\left[\frac{D\left(\dfrac{\partial^2\psi_1}{\partial y^2},\ \psi_0\right)}{D(x,\ y)} + \frac{D\left(\dfrac{\partial^2\psi_0}{\partial y^2},\ \psi_1\right)}{D(x,\ y)}\right] - 2\,\frac{\partial^4\psi_0}{\partial x^2\partial y^2} \tag{15}$$

and the boundary conditions

$$\left.\frac{\partial^2\psi_2}{\partial y^2}\right|_{y=f(x)} = 0;\ \left.\frac{\partial\psi_2}{\partial y}\right|_{y=0} = 0;\ \left.\frac{\partial\psi_2}{\partial x}\right|_{y=0} = 0;\ \psi_2[x,\ f(x)] - \psi_2[x,\ 0] = 0; \tag{16}$$

yielding

$$\psi_2 = \frac{C_7}{6}\, y^3 + \frac{C_8}{2}\, y^2 + C_9 + \Pi_2(x,\ y), \tag{17}$$

where

$$C_7 = \frac{\mathrm{Re}^2 w}{320}\left[39w\,(-c - q + q\eta) - \frac{3}{14}\,(114c^2 + 197cq + 139q^2)\,\eta_x\right]$$

$$+ 3\left[\frac{\mathrm{Re}^2}{121275}(1404q^3 + 1896cq^2 + 889c^2q - 166c^3) - \frac{1}{10}\,(6c + 19q)\right]\eta_{xx} + O(\eta^4);$$

$\Pi_2(x, y)$ is the particular solution to the nonhomogeneous Eq. (15) (not shown here because of its unwieldiness), $\eta_{xx} \equiv d^2\eta/dx^2$, and C_9 is an arbitrary constant; C_8 is not needed for further calculations.

Thus, we have a solution to the auxiliary problem in terms of ψ as a function of the yet unknown quantity f(x) and, in order to determine the latter, we need now the boundary condition (5). Differentiating Eq. (5) along the free surface, we can obtain

$$\left(\frac{\partial P}{\partial x}+\frac{\partial P}{\partial y}f_x+\frac{2}{\mathrm{Re}}\frac{\partial^3\psi_0}{\partial x^2\partial y}\right)\Big|_{y=f(x)}=0, \tag{18}$$

within the necessary accuracy, where $\partial P/\partial x$ and $\partial P/\partial y$ from Eqs. (1) and (2) respectively are

$$\frac{\partial P}{\partial x}=\frac{1}{\mathrm{Re}}\left(C_1+C_4+C_7-\frac{\partial^3\psi_0}{\partial x^2\partial y}\right)+\frac{1}{\mathrm{Fr}^2}\sin\alpha+O(\varepsilon^3);$$

$$\frac{\partial P}{\partial y}=-\frac{1}{\mathrm{Fr}^2}\cos\alpha+O(\varepsilon^2). \tag{19}$$

Inserting (19) into (18), with the values of the constants C_1, C_4, and C_7 just determined and also considering that $q=1-c$, we obtain a second-order ordinary differential equation in $\eta(x)$:

$$\eta_{xx}+A_1\eta_x'+A_2\eta\eta_x'+A_3\eta+A_4\eta^2+A_5=0, \tag{20}$$

where

$$A_1=\frac{\mathrm{Re}}{B}\left[\frac{2}{35}(7c^2-17c+9)-\frac{\mathrm{Re}\,w}{4480}(56c^2-81c+139)-\frac{1}{3}\frac{1}{\mathrm{Fr}^2}\cos\alpha\right];$$

$$A_2=\frac{2}{35}\frac{\mathrm{Re}}{B}(-24c^2+52c-27);\ A_3=\frac{1}{B}\Big\{3-c-\frac{3}{8}\mathrm{Re}\,w$$

$$\times\left[c-2-\frac{13}{120}\mathrm{Re}\,w(1-c)\right]\Big\};\ A_4=\frac{3}{B}\left[c-2-\frac{1}{8}\mathrm{Re}\,w(3-2c)\right];$$

$$A_5=\frac{1}{B}\left[\frac{1}{3}\mathrm{Re}\frac{1}{\mathrm{Fr}^2}\sin\alpha-1-\frac{3}{8}\mathrm{Re}\,w\left(1+\frac{13}{120}\mathrm{Re}\,w\right)\right];$$

$$B=\frac{3}{5}(3c-4)+\frac{\mathrm{Re}^2}{121275}(231c^3-1309c^2+2316c-1404).$$

The parameters in the periodic solution to equations of the (20) kind can, for small values of the Reynolds number, be expressed in terms of the coefficients in Eq. (20) [13]. It has been shown in [11, 13] that for the undular mode in the case of a thin viscous liquid film the dimensionless velocity c is almost three times higher than the mean velocity of plane-parallel flow, from which follows that $A_1<0$, $A_2>0$, $A_3>0$, and $A_4>0$. Considering this, we have the following limits for the injection velocity:

$$0<w\ll\frac{1}{\mathrm{Re}}, \tag{21}$$

and this range should be regarded as the validity criterion for applying the method of "narrow" bands to the given flow problem.

With the Reynolds number Re of the film flow, the inclination angle α of the wall, and the injection velocity w of the liquid given according to (21), the solution to Eq. (20) will relate the shape and the wavelength of the profile to the injection velocity w. Based on calculations, the shape of the wave profile as a function of the parameter w is shown in Fig. 2 and the wavelength as a function of the parameter w is shown in Fig. 3. We note that a higher injection velocity, as is evident from Figs. 2 and 3, will result in a smaller wave amplitude and a shorter wavelength on the film surface with the film becoming thicker.

NOTATION

- w is the injection velocity;
- U is the wave velocity;
- Q is the flow rate;
- H is the depth of liquid film;
- x, y is the dimensionless cartesian coordinates;
- Re is the Reynolds number;
- ν is the kinematic viscosity;
- Fr is the Froude number;
- g is the acceleration of gravity;
- c is the dimensionless wave velocity;
- ψ is the flow function;

u, v	are the velocity components along x and y respectively;
P	is the pressure;
q	is the dimensionless flow rate;
w	is the dimensionless injection velocity;
ε	is the small parameter;
$C_1, C_2, \ldots, C_9$	are the unknown functions;
$A_1, A_2, \ldots, A_5$	are the constants;
Π_1, Π_2	are the particular solutions to nonhomogeneous equations;
λ	is the wavelength;
α	is the inclination angle of porous wall;
$f(x), \eta(x)$	are the shape functions of the undular film surface.

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